

HOPF ALGEBRAS AND POLYNOMIAL IDENTITIES

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ABSTRACT. This is a survey of results obtained jointly with E. Aljadeff and published in [2]. We explain how to set up a theory of polynomial identities for comodule algebras over a Hopf algebra, and concentrate on the universal comodule algebra constructed from the identities satisfied by a given comodule algebra. All concepts are illustrated with various examples.

KEY WORDS: Polynomial identity, Hopf algebra, comodule, localization

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INTRODUCTION

As has been stressed many times (see, e.g., [19]), Hopf Galois extensions can be viewed as non-commutative analogues of principal fiber bundles (also known as G -torsors), where the role of the structural group is played by a Hopf algebra. Such extensions abound in the world of quantum groups and of non-commutative geometry. The problem of constructing systematically all Hopf Galois extensions of a given algebra for a given Hopf algebra and of classifying them up to isomorphism has been addressed in a number of papers, such as [4, 7, 9, 12, 13, 14, 15, 18] to quote but a few.

A new approach to the classification problem of Hopf Galois extensions was recently advanced by Eli Aljadeff and the present author in [2]; this approach uses classical techniques from non-commutative algebra such as *polynomial identities* (such techniques had previously been used in [1] for group-graded algebras). In [2] we developed a theory of identities for any comodule algebra over a given Hopf algebra H , hence for any Hopf Galois extension. As a result, out of the identities for an H -comodule algebra A , we obtained a *universal H -comodule algebra* $\mathcal{U}_H(A)$. It turns out that if A is a cleft H -Galois object (i.e., a comodule algebra obtained from H by twisting its product with the help of a two-cocycle) with trivial center, then a suitable central localization of $\mathcal{U}_H(A)$ is an H -Galois extension of its center. We thus obtain a “non-commutative principal fiber bundle” whose base space is the spectrum of some localization of the center of $\mathcal{U}_H(A)$.

This survey is organized as follows. After a preliminary section on comodule algebras, we define the concept of an H -identity for such algebras in § 2. We illustrate this concept with a few examples and we attach a universal H -comodule algebra $\mathcal{U}_H(A)$ to each H -comodule algebra A .

In § 3 turning to the special case where $A = {}^{\alpha}H$ is a twisted comodule algebra, we exhibit a universal comodule algebra map that allows us to detect the H -identities for A .

In § 4 we construct a commutative domain \mathcal{B}_H^α and we state that under some natural extra condition, \mathcal{B}_H^α is the center of a suitable central localization of $\mathcal{U}_H(A)$; moreover after localization, $\mathcal{U}_H(A)$ becomes a free module over its center.

Lastly in § 5, we illustrate all previous constructions with the help of the four-dimensional Sweedler algebra, thus giving complete answers in this simple, but non-trivial example. We end the paper with an open question on Taft algebras.

The material of the present text is mainly taken from [2], for which it provides an easy access. The reader is advised to complement it with [10, 11].

1. HOPF ALGEBRAS AND COACTIONS

1.1. Standing assumption. We fix a field k over which all our constructions are defined. In particular, all linear maps are supposed to be k -linear and unadorned tensor products mean tensor products over k . Throughout the survey we assume that the ground field k is *infinite*.

By algebra we always mean an associative unital k -algebra. We suppose the reader familiar with the language of Hopf algebra, as expounded for instance in [20]. As is customary, we denote the coproduct of a Hopf algebra by Δ , its counit by ε , and its antipode by S . We also make use of a Heyneman-Sweedler-type notation for the image

$$\Delta(x) = x_1 \otimes x_2$$

of an element x of a Hopf algebra H under the coproduct, and we write

$$\Delta^{(2)}(x) = x_1 \otimes x_2 \otimes x_3$$

for the iterated coproduct $\Delta^{(2)} = (\Delta \otimes \text{id}_H) \circ \Delta = (\text{id}_H \otimes \Delta) \circ \Delta$, and so on.

1.2. Comodule algebras. Let H be a Hopf algebra. Recall that an H -comodule algebra is an algebra A equipped with a right H -comodule structure whose (coassociative, counital) *coaction*

$$\delta : A \rightarrow A \otimes H$$

is an algebra map. The subalgebra A^H of *coinvariants* of an H -comodule algebra A is defined by

$$A^H = \{a \in A \mid \delta(a) = a \otimes 1\}.$$

Given two H -comodule algebras A and A' with respective coactions δ and δ' , an algebra map $f : A \rightarrow A'$ is an H -comodule algebra map if

$$\delta' \circ f = (f \otimes \text{id}_H) \circ \delta.$$

We denote by Alg^H the category whose objects are H -comodule algebras and arrows are H -comodule algebra maps.

Let us give a few examples of comodule algebras.

Example 1.1. If $H = k$, then an H -comodule algebra is nothing but an ordinary (associative, unital) algebra.

Example 1.2. The algebra $H = k[G]$ of a group G is a Hopf algebra with coproduct, counit, and antipode given for all $g \in G$ by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}.$$

It is well-known (see [5, Lemma 4.8]) that an H -comodule algebra A is the same as a G -graded algebra

$$A = \bigoplus_{g \in G} A_g, \quad A_g A_h \subset A_{gh}.$$

The coaction $\delta : A \rightarrow A \otimes H$ is given by $\delta(a) = a \otimes g$ for all $a \in A_g$ and $g \in G$. We have $A^H = A_e$, where e is the neutral element of G .

Example 1.3. Let G be a finite group and $H = k^G$ be the algebra of k -valued functions on a finite group G . This algebra can be equipped with a Hopf algebra structure that is dual to the Hopf algebra $k[G]$ above. An H -comodule algebra A is the same as a G -algebra, i.e., an algebra equipped with a left action of G on A by group automorphisms.

If we denote the action of $g \in G$ on $a \in A$ by ${}^g a$, then the coaction $\delta : A \rightarrow A \otimes H$ is given by

$$\delta(a) = \sum_{g \in G} {}^g a \otimes e_g,$$

where $\{e_g\}_{g \in G}$ is the basis of H consisting of the functions e_g defined by $e_g(h) = 1$ if $h = g$, and 0 otherwise.

The subalgebra of coinvariants of A coincides with the subalgebra of G -invariant elements: $A^H = A^G$.

Example 1.4. Any Hopf algebra H is an H -comodule algebra whose coaction coincides with the coproduct of H :

$$\delta = \Delta : H \rightarrow H \otimes H.$$

In this case the coinvariants of H are exactly the scalar multiples of the unit of H ; in other words, $H^H = k1$.

2. IDENTITIES

2.1. Polynomial identities. Let A be an algebra. A *polynomial identity* for an algebra A is a polynomial $P(X, Y, Z, \dots)$ in a finite number of non-commutative variables X, Y, Z, \dots such that

$$P(x, y, z, \dots) = 0$$

for all $x, y, z, \dots \in A$.

Examples 2.1. (a) The polynomial $XY - YX$ is a polynomial identity for any commutative algebra.

(b) If $A = M_2(k)$ is the algebra of 2×2 -matrices with entries in k , then

$$(XY - YX)^2 Z - Z(XY - YX)^2$$

is a polynomial identity for A . (Use the Cayley-Hamilton theorem to check this.)

The concept of a polynomial identity first emerged in the 1920's in an article [6] on the foundation of projective geometry by Max Dehn, the topologist. The above polynomial identity for the algebra of 2×2 -matrices appeared in 1937 in [22]. Today there is an abundant literature on polynomial identities; see for instance [8, 17].

For algebras graded by a group G there exists the concept of a graded polynomial identity (see [1, 3]). In this case we need to take a family of non-commutative variables X_g, Y_g, Z_g, \dots for each element $g \in G$. Given a G -graded algebra $A = \bigoplus_{g \in G} A_g$, a *graded polynomial identity* is a polynomial P in these indexed variables such that P vanishes upon any substitution of each variable X_g appearing in P by an element of the g -component A_g .

In general, we should keep in mind that in order to define polynomial identities for a class of algebras, we need to single out

- (i) a suitable algebra of non-commutative polynomials and
- (ii) a suitable notion of specialization for these polynomials.

The algebras of interest to us in this survey are comodule algebras over a Hopf algebra H . The non-commutative variables we wish to use will be indexed by the elements of some linear basis of H . Since in general a Hopf algebra does not have a natural basis, we find it preferable to use a more canonical construction, namely the tensor algebra over H , and to resort to a given basis only when we need to perform computations.

2.2. Definition and examples of H -identities. Let H be a Hopf algebra. We pick a copy X_H of the underlying vector space of H and we denote the identity map from H to X_H by $x \mapsto X_x$ for all $x \in H$.

Consider the *tensor algebra* $T(X_H)$ of the vector space X_H over the ground field k :

$$T(X_H) = \bigoplus_{r \geq 0} T^r(X_H),$$

where $T^r(X_H) = X_H^{\otimes r}$ is the tensor product of r copies of X_H over k , with the convention $T^0(X_H) = k$. If $\{x_i\}_{i \in I}$ is some linear basis of H , then $T(X_H)$ is isomorphic to the algebra of non-commutative polynomials in the indeterminates X_{x_i} ($i \in I$).

Beware that the product $X_x X_y$ of symbols in the tensor algebra is different from the symbol X_{xy} attached to the product of x and y in H ; the former is of degree 2 while the latter is of degree 1.

The algebra $T(X_H)$ is an H -comodule algebra equipped with the coaction

$$\delta : T(X_H) \rightarrow T(X_H) \otimes H ; \quad X_x \mapsto X_{x_1} \otimes x_2.$$

Note that $T(X_H)$ is *graded* with all generators X_x in degree 1. The coaction preserves the grading, where $T(X_H) \otimes H$ is graded by

$$(T(X_H) \otimes H)_r = T^r(X_H) \otimes H$$

for all $r \geq 0$.

We now give the main definition of this section.

Definition 2.2. Let A be an H -comodule algebra. An element $P \in T(X_H)$ is an H -identity for A if $\mu(P) = 0$ for all H -comodule algebra maps

$$\mu : T(X_H) \rightarrow A.$$

To convey the feeling of what an H -identity is, let us give some simple examples.

Example 2.3. Let $H = k$ be the one-dimension Hopf algebra as in Example 1.1. An H -comodule algebra A is then the same as an algebra. In this case, $T(X_H)$ coincides with the polynomial algebra $k[X_1]$ and an H -comodule algebra map is nothing but an algebra map. Therefore, an element $P(X_1) \in T(X_H) = k[X_1]$ is an H -identity for A if and only if all $P(a) = 0$ for all $a \in A$. Since k is assumed to be infinite, it follows that there are no non-zero H -identities for A .

Example 2.4. Let $H = k[G]$ be a group Hopf algebra as in Example 1.2. We know that an H -comodule algebra is a G -graded algebra $A = \bigoplus_{g \in G} A_g$. Since $\{g\}_{g \in G}$ is a basis of H , the tensor algebra $T(X_H)$ is the algebra of non-commutative polynomials in the indeterminates X_g ($g \in G$).

It is easy to check that an algebra map $\mu : T(X_H) \rightarrow A$ is an H -comodule algebra map if and only if $\mu(X_g) \in A_g$ for all $g \in G$. This remark allows us to produce the following examples of H -identities.

- (a) Suppose that A is *trivially graded*, i.e., $A_g = 0$ for all $g \neq e$. Then any non-commutative polynomial in the indeterminates X_g with $g \neq e$ is killed by any H -comodule algebra map $\mu : T(X_H) \rightarrow A$. Therefore, such a polynomial is an H -identity for A .
- (b) Suppose that the trivial component A_e is *central* in A . We claim that

$$X_g X_{g^{-1}} X_h - X_h X_g X_{g^{-1}}$$

is an H -identity for A for all $g, h \in G$. Indeed, for any H -comodule algebra map $\mu : T(X_H) \rightarrow A$, we have

$$\mu(X_g) \in A_g \quad \text{and} \quad \mu(X_{g^{-1}}) \in A_{g^{-1}};$$

therefore, $\mu(X_g X_{g^{-1}}) = \mu(X_g) \mu(X_{g^{-1}})$ belongs to A_e , hence commutes with $\mu(X_h)$. One shows in a similar fashion that if g is an element of G of finite order N , then for all $h \in G$,

$$X_g^N X_h - X_h X_g^N$$

is an H -identity for A .

Example 2.5. Let H be an arbitrary Hopf algebra, and let A be an H -comodule algebra such that the subalgebra A^H of coinvariants is central in A (the twisted comodule algebras of § 3.1 satisfy the latter condition).

For $x, y \in H$ consider the following elements of $T(X_H)$:

$$P_x = X_{x_1} X_{S(x_2)} \quad \text{and} \quad Q_{x,y} = X_{x_1} X_{y_1} X_{S(x_2 y_2)}.$$

Then for all $x, y, z \in H$,

$$P_x X_z - X_z P_x \quad \text{and} \quad Q_{x,y} X_z - X_z Q_{x,y}$$

are H -identities for A . Indeed, P_x and $Q_{x,y}$ are coinvariant elements of $T(X_H)$; see [2, Lemma 2.1]. It follows that for any H -comodule algebra map $\mu : T(X_H) \rightarrow A$, the elements $\mu(P_x)$ and $\mu(Q_{x,y})$ are coinvariant, hence central, in A .

More sophisticated examples of H -identities will be given in § 5.

2.3. The ideal of H -identities. Let H be a Hopf algebra and A an H -comodule algebra. Denote the set of all H -identities for A by $I_H(A)$. By definition,

$$I_H(A) = \bigcap_{\mu \in \text{Alg}^H(T(X_H), A)} \text{Ker } \mu.$$

A proof of the following assertions can be found in [2, Prop. 2.2].

Proposition 2.6. *The set $I_H(A)$ has the following properties:*

(a) *it is a graded ideal of $T(X_H)$, i.e.,*

$$I_H(A)T(X_H) \subset I_H(A) \supset T(X_H)I_H(A)$$

and

$$I_H(A) = \bigoplus_{r \geq 0} \left(I_H(A) \cap T^r(X_H) \right);$$

(b) *it is a right H -coideal of $T(X_H)$, i.e.,*

$$\delta(I_H(A)) \subset I_H(A) \otimes H.$$

Note that for any H -comodule algebra map $\mu : T(X_H) \rightarrow A$, we have $\mu(1) = 1$; therefore, the degree 0 component of $I_H(A)$ is always trivial:

$$I_H(A) \cap T^0(X_H) = 0.$$

If, in addition, there exists an injective H -comodule map $H \rightarrow A$, then the degree 1 component of $I_H(A)$ is also trivial:

$$I_H(A) \cap T^1(X_H) = 0.$$

Remark 2.7. Right from the beginning we required the ground field k to be infinite. This assumption is used for instance to establish that $I_H(A)$ is a graded ideal of $T(X_H)$. Let us give a proof of this fact in order to show how the assumption is used. Indeed, expand $P \in I_H(A)$ as

$$P = \sum_{r \geq 0} P_r$$

with $P_r \in T^r(X_H)$ for all $r \geq 0$. To prove that $I_H(A)$ is a graded ideal, it suffices to check that each P_r is in $I_H(A)$. Given a scalar $\lambda \in k$, consider the algebra endomorphism λ_* of $T(X_H)$ defined by $\lambda(X_x) = \lambda X_x$ for all $x \in H$; clearly, λ_* is an H -comodule map. If $\mu : T(X_H) \rightarrow A$ is an H -comodule algebra map, then so is $\mu \circ \lambda_*$. Since $P \in I_H(A)$, we have

$$\sum_{r \geq 0} \lambda^r \mu(P_r) = (\mu \circ \lambda_*)(P) = 0.$$

The A -valued polynomial $\sum_{r \geq 0} \lambda^r \mu(P_r)$ takes zero values for all $\lambda \in k$. By the assumption on k , this implies that its coefficients are all zero, i.e., $\mu(P_r) = 0$ for all $r \geq 0$. Since this holds for all $\mu \in \text{Alg}^H(T(X_H), A)$, we obtain $P_r \in I_H(A)$ for all $r \geq 0$.

If the ground field is *finite*, then Definition 2.2 still makes sense, but the ideal $I_H(A)$ may no longer be graded. Indeed, let k be the finite field \mathbb{F}_p and $H = k$. Then for $q = p^N$, the finite field \mathbb{F}_q is an H -comodule algebra. In view of Example 2.3, the polynomial $X_1^q - X_1$ is an H -identity for \mathbb{F}_q , but clearly the homogeneous summands in this polynomial, namely X_1^q and X_1 , are not H -identities.

2.4. The universal H -comodule algebra. Let A be an H -comodule algebra and $I_H(A)$ the ideal of H -identities for A defined above. Since $I_H(A)$ is a graded ideal of $T(X_H)$, we may consider the quotient algebra

$$\mathcal{U}_H(A) = T(X_H)/I_H(A).$$

The grading on $T(X_H)$ induces a grading on $\mathcal{U}_H(A)$. As $I_H(A)$ is a right H -coideal of $T(X_H)$, the quotient algebra $\mathcal{U}_H(A)$ carries an H -comodule algebra structure inherited from $T(X_H)$.

By definition of $\mathcal{U}_H(A)$, all H -identities for A vanish in $\mathcal{U}_H(A)$. For this reason we call $\mathcal{U}_H(A)$ the *universal H -comodule algebra attached to A* .

The algebra $\mathcal{U}_H(A)$ has two interesting subalgebras:

- (i) The subalgebra $\mathcal{U}_H(A)^H$ of *coinvariants* of $\mathcal{U}_H(A)$.
- (ii) The *center* $\mathcal{Z}_H(A)$ of $\mathcal{U}_H(A)$.

We now raise the following question. Suppose that the comodule algebra A is free as a module over the subalgebra of coinvariants A^H (or over its center); is $\mathcal{U}_H(A)$, or rather some suitable central localization of it, then free as a module over some localization of $\mathcal{U}_H(A)^H$ (or of $\mathcal{Z}_H(A)$)? An answer to this question will be given below (see Theorem 4.5) for a special class of comodule algebras, which we introduce in the next section.

3. DETECTING H -IDENTITIES

Fix a Hopf algebra H . We now define a special class of H -comodule algebras for which we can detect all H -identities.

3.1. Twisted comodule algebras. Recall that a *two-cocycle* α on H is a bilinear form $\alpha : H \times H \rightarrow k$ such that

$$\alpha(x_1, y_1) \alpha(x_2 y_2, z) = \alpha(y_1, z_1) \alpha(x, y_2 z_2)$$

for all $x, y, z \in H$. We assume that α is convolution-invertible and write α^{-1} for its inverse. For simplicity, we also assume that α is normalized, i.e.,

$$\alpha(x, 1) = \alpha(1, x) = \varepsilon(x)$$

for all $x \in H$.

Any Hopf algebra possesses at least one normalized convolution-invertible two-cocycle, namely the *trivial* two-cocycle α_0 , which is defined by

$$\alpha_0(x, y) = \varepsilon(x) \varepsilon(y)$$

for all $x, y \in H$.

Let u_H be a copy of the underlying vector space of H . Denote the identity map from H to u_H by $x \mapsto u_x$ ($x \in H$). We define the *twisted algebra* ${}^\alpha H$ as the vector space u_H equipped with the associative product given by

$$u_x u_y = \alpha(x_1, y_1) u_{x_2 y_2}$$

for all $x, y \in H$. This product is associative because of the above cocycle condition; the two-cocycle α being normalized, u_1 is the unit of ${}^\alpha H$.

The algebra ${}^\alpha H$ is an H -comodule algebra with coaction $\delta : {}^\alpha H \rightarrow {}^\alpha H \otimes H$ given for all $x \in H$ by

$$\delta(u_x) = u_{x_1} \otimes x_2.$$

It is easy to check that the subalgebra of coinvariants of ${}^\alpha H$ coincides with $k u_1$, which lies in the center of ${}^\alpha H$.

Note that if $\alpha = \alpha_0$ is the trivial two-cocycle, then ${}^\alpha H = H$ is the H -comodule algebra of Example 1.4.

The twisted comodule algebras of the form ${}^\alpha H$ coincide with the so-called *cleft H -Galois objects*; see [16, Prop. 7.2.3]. It is therefore an important class of comodule algebras. We next show how we can detect H -identities for such comodule algebras.

3.2. The universal comodule algebra map. We pick a third copy t_H of the underlying vector space of H and denote the identity map from H to t_H by $x \mapsto t_x$ ($x \in H$). Let $S(t_H)$ be the *symmetric algebra* over the vector space t_H . If $\{x_i\}_{i \in I}$ is a linear basis of H , then $S(t_H)$ is isomorphic to the (commutative) algebra of polynomials in the indeterminates t_{x_i} ($i \in I$).

We consider the algebra $S(t_H) \otimes {}^\alpha H$. As a k -algebra, it is generated by the symbols $t_z u_x$ ($x, z \in H$) (we drop the tensor product sign \otimes between the t -symbols and the u -symbols).

The algebra $S(t_H) \otimes {}^\alpha H$ is an H -comodule algebra whose $S(t_H)$ -linear coaction extends the coaction of ${}^\alpha H$:

$$\delta(t_z u_x) = t_z u_{x_1} \otimes x_2.$$

Define an algebra map $\mu_\alpha : T(X_H) \rightarrow S(t_H) \otimes {}^\alpha H$ by

$$\mu_\alpha(X_x) = t_{x_1} u_{x_2}$$

for all $x \in H$. The map μ_α possesses the following properties (see [2, Sect. 4]).

Proposition 3.1. (a) *The map $\mu_\alpha : T(X_H) \rightarrow S(t_H) \otimes {}^\alpha H$ is an H -comodule algebra map.*

(b) *For every H -comodule algebra map $\mu : T(X_H) \rightarrow {}^\alpha H$, there is a unique algebra map $\chi : S(t_H) \rightarrow k$ such that*

$$\mu = (\chi \otimes \text{id}) \circ \mu_\alpha.$$

In other words, any H -comodule algebra map $\mu : T(X_H) \rightarrow {}^\alpha H$ can be obtained from μ_α by specialization. For this reason we call μ_α the *universal comodule algebra map* for ${}^\alpha H$.

Theorem 3.2. *An element $P \in T(X_H)$ is an H -identity for ${}^\alpha H$ if and only if $\mu_\alpha(P) = 0$; equivalently,*

$$I_H({}^\alpha H) = \ker(\mu_\alpha).$$

This result is a consequence of Proposition 3.1. It allows us to detect the H -identities for any twisted comodule algebra: it suffices to check them in the easily controllable algebra $S(t_H) \otimes {}^\alpha H$. In § 5 we shall show how to apply this result in an interesting example.

Let us derive some consequences of Theorem 3.2. To simplify notation, we denote the ideal of H -identities $I_H({}^\alpha H)$ by I_H^α , the universal H -comodule algebra $\mathcal{U}_H({}^\alpha H)$ by \mathcal{U}_H^α , and the center $\mathcal{Z}_H({}^\alpha H)$ of \mathcal{U}_H^α by \mathcal{Z}_H^α .

Corollary 3.3. (a) *The map $\mu_\alpha : T(X_H) \rightarrow S(t_H) \otimes {}^\alpha H$ induces an injection of comodule algebras*

$$\bar{\mu}_\alpha : \mathcal{U}_H^\alpha \hookrightarrow S(t_H) \otimes {}^\alpha H.$$

(b) An element of \mathcal{U}_H^α belongs to the subalgebra $(\mathcal{U}_H^\alpha)^H$ of coinvariants if and only if its image under $\bar{\mu}_\alpha$ sits in the subalgebra $S(t_H) \otimes u_1$.

We also proved that an element of \mathcal{U}_H^α belongs to the center \mathcal{Z}_H^α if and only if its image under $\bar{\mu}_\alpha$ sits in the subalgebra $S(t_H) \otimes Z({}^\alpha H)$, where $Z({}^\alpha H)$ is the center of ${}^\alpha H$ (see [2, Prop. 8.2]). In particular, since u_1 is central in ${}^\alpha H$, it follows that all coinvariant elements of \mathcal{U}_H^α belong to the center \mathcal{Z}_H^α .

We mention another consequence: it asserts that there always exist non-zero H -identities for any non-trivial finite-dimensional twisted comodule algebra.

Corollary 3.4. *If $2 \leq \dim_k H < \infty$, then $I_H^\alpha \neq \{0\}$.*

Proof. Suppose that $I_H^\alpha = \{0\}$. Then in view of $\mathcal{U}_H^\alpha = T(X_H)/I_H^\alpha$ and of Corollary 3.3, we would have an injective linear map

$$T^r(X_H) \hookrightarrow S^r(X_H) \otimes {}^\alpha H$$

for all $r \geq 0$. (Here $S^r(X_H)$ is the subspace of elements of degree r in $S(t_H)$.) Taking dimensions and setting $\dim_k H = n$, we would obtain the inequality

$$n^r \leq n \binom{r+n-1}{n-1},$$

which is impossible for large r . \square

4. LOCALIZING THE UNIVERSAL COMODULE ALGEBRA

We now wish to address the question raised in § 2.4 in the case A is a twisted comodule algebra of the form ${}^\alpha H$, where H is a Hopf algebra and α is a normalized convolution-invertible two-cocycle on H .

4.1. The generic base algebra. Recall the symmetric algebra $S(t_H)$ introduced in § 3.2. By [2, Lemma A.1] there is a unique linear map $x \mapsto t_x^{-1}$ from H to the field of fractions $\text{Frac } S(t_H)$ of $S(t_H)$ such that for all $x \in H$,

$$\sum_{(x)} t_{x(1)} t_{x(2)}^{-1} = \sum_{(x)} t_{x(1)}^{-1} t_{x(2)} = \varepsilon(x) 1.$$

(The algebra of fractions generated by the elements t_x and t_x^{-1} ($x \in H$) is Takeuchi's free commutative Hopf algebra on the coalgebra underlying H ; see [21].)

Examples 4.1. (a) If g is a *group-like* element, i.e., $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, then

$$t_g^{-1} = \frac{1}{t_g}.$$

(b) If x is a *skew-primitive* element, i.e., $\Delta(x) = g \otimes x + x \otimes h$ for some group-like elements g, h , then

$$t_x^{-1} = -\frac{t_x}{t_g t_h}.$$

For $x, y \in H$, define the following elements of the fraction field $\text{Frac } S(t_H)$:

$$\sigma(x, y) = \sum_{(x), (y)} t_{x(1)} t_{y(1)} \alpha(x(2), y(2)) t_{x(3)y(3)}^{-1}$$

and

$$\sigma^{-1}(x, y) = \sum_{(x), (y)} t_{x(1)y(1)} \alpha^{-1}(x(2), y(2)) t_{x(3)}^{-1} t_{y(3)}^{-1},$$

where α^{-1} is the inverse of α .

The map $(x, y) \in H \times H \mapsto \sigma(x, y) \in \text{Frac } S(t_H)$ is a two-cocycle with values in the fraction field $\text{Frac } S(t_H)$.

Definition 4.2. *The generic base algebra is the subalgebra \mathcal{B}_H^α of $\text{Frac } S(t_H)$ generated by the elements $\sigma(x, y)$ and $\sigma^{-1}(x, y)$, where x and y run over H .*

Since \mathcal{B}_H^α is a subalgebra of the field $\text{Frac } S(t_H)$, it is a domain and the Krull dimension of \mathcal{B}_H^α cannot exceed the Krull dimension of $S(t_H)$, which is $\dim_k H$. Actually, it is proved in [11, Cor. 3.7] that if the Hopf algebra H is finite-dimensional, then the Krull dimension of \mathcal{B}_H^α is exactly equal to $\dim_k H$. More properties of the generic base algebra are given in [11].

Example 4.3. If $H = k[G]$ is the Hopf algebra of a group G and $\alpha = \alpha_0$ is the trivial two-cocycle, then the generic base algebra \mathcal{B}_H^α is the algebra generated by the Laurent polynomials

$$\left(\frac{t_g t_h}{t_{gh}} \right)^{\pm 1},$$

where g, h run over G . A complete computation for the (in)finite cyclic groups $G = \mathbb{Z}$ and $G = \mathbb{Z}/N$ was given in [10, Sect. 3.3].

4.2. Non-degenerate cocycles. We now restrict to the case when α is a *non-degenerate* two-cocycle, i.e., when the center of the twisted algebra ${}^\alpha H$ is one-dimensional. In this case, the center of ${}^\alpha H$ coincides with the subalgebra of coinvariants.

Recall the injective algebra map $\bar{\mu}_\alpha : \mathcal{U}_H^\alpha \rightarrow S(t_H) \otimes {}^\alpha H$ of Corollary 3.3. By this corollary and the subsequent comment, it follows that in the non-degenerate case the center \mathcal{Z}_H^α of \mathcal{U}_H^α coincides with the subalgebra $(\mathcal{U}_H^\alpha)^H$ of coinvariants, and we have

$$\mathcal{Z}_H^\alpha = (\mathcal{U}_H^\alpha)^H = \bar{\mu}_\alpha^{-1}(S(t_H) \otimes u_1).$$

The following result connects \mathcal{Z}_H^α to the generic base algebra \mathcal{B}_H^α introduced in § 4.1 (see [2, Prop. 9.1]).

Proposition 4.4. *If α is a non-degenerate two-cocycle on H , then $\bar{\mu}_\alpha$ maps \mathcal{Z}_H^α into $\mathcal{B}_H^\alpha \otimes u_1$.*

This result allows us to view the center \mathcal{Z}_H^α of \mathcal{U}_H^α as a subalgebra of the generic base algebra \mathcal{B}_H^α . It follows from the discussion in § 4.1 that \mathcal{Z}_H^α is a domain whose Krull dimension is at most $\dim_k H$.

We may now consider the \mathcal{B}_H^α -algebra

$$\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha.$$

It is an H' -comodule algebra, where $H' = \mathcal{B}_H^\alpha \otimes H$.

The following answers the question raised in § 2.4.

Theorem 4.5. *If H is a Hopf algebra and α is a non-degenerate two-cocycle on H such that \mathcal{B}_H^α is a localization of \mathcal{Z}_H^α , then $\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha$ is a cleft H -Galois extension of \mathcal{B}_H^α . In particular, there is a comodule isomorphism*

$$\mathcal{B}_H^\alpha \otimes_{\mathcal{Z}_H^\alpha} \mathcal{U}_H^\alpha \cong \mathcal{B}_H^\alpha \otimes H.$$

It follows that under the hypotheses of the theorem, a suitable central localization of the universal comodule algebra \mathcal{U}_H^α is free of rank $\dim_k H$ as a module over its center.

5. AN EXAMPLE: THE SWEEDLER ALGEBRA

We assume in this section that the characteristic of k is different from 2.

5.1. Presentation and twisted comodule algebras. The *Sweedler algebra* H_4 is the algebra generated by two elements x, y subject to the relations

$$x^2 = 1, \quad xy + yx = 0, \quad y^2 = 0.$$

It is four-dimensional. As a basis of H_4 , we take the set $\{1, x, y, z\}$, where $z = xy$.

The algebra H_4 carries the structure of a non-commutative, non-cocommutative Hopf algebra with coproduct, counit, and antipode given by

$$\begin{aligned} \Delta(1) &= 1 \otimes 1, & \Delta(x) &= x \otimes x, \\ \Delta(y) &= 1 \otimes y + y \otimes x, & \Delta(z) &= x \otimes z + z \otimes 1, \\ \varepsilon(1) &= \varepsilon(x) = 1, & \varepsilon(y) &= \varepsilon(z) = 0, \\ S(1) &= 1, & S(x) &= x, \\ S(y) &= z, & S(z) &= -y. \end{aligned}$$

The tensor algebra $T(H_4)$ is the free non-commutative algebra on the four symbols

$$E = X_1, \quad X = X_x, \quad Y = X_y, \quad Z = X_z,$$

whereas $S(t_{H_4})$ is the polynomial algebra on the symbols t_1, t_x, t_y, t_z .

Masuoka [13] (see also [7]) showed that any twisted H_4 -comodule algebra as in § 3.1 has, up to isomorphism, the following presentation:

$${}^\alpha H_4 = k \langle u_x, u_y \mid u_x^2 = au_1, \quad u_x u_y + u_y u_x = bu_1, \quad u_y^2 = cu_1 \rangle$$

for some scalars a, b, c with $a \neq 0$. To indicate the dependence on the parameters a, b, c , we denote ${}^\alpha H_4$ by $A_{a,b,c}$.

The center of $A_{a,b,c}$ consists of the scalar multiples of the unit u_1 for all values of a, b, c . In other words, all two-cocycles on H_4 are non-degenerate.

The coaction $\delta : A_{a,b,c} \rightarrow A_{a,b,c} \otimes H_4$ is determined by

$$\delta(u_x) = u_x \otimes x \quad \text{and} \quad \delta(u_y) = u_1 \otimes y + u_y \otimes x.$$

As observed in § 3.1, the coinvariants of $A_{a,b,c}$ consists of the scalar multiples of the unit u_1 . Therefore, coinvariants and central elements of $A_{a,b,c}$ coincide.

5.2. Identities. In this situation, the universal comodule algebra map

$$\mu_\alpha : T(X_H) \rightarrow S(t_H) \otimes A_{a,b,c}$$

is given by

$$\begin{aligned} \mu_\alpha(E) &= t_1 u_1, & \mu_\alpha(X) &= t_x u_x, \\ \mu_\alpha(Y) &= t_1 u_y + t_y u_x, & \mu_\alpha(Z) &= t_x u_z + t_z u_1. \end{aligned}$$

Let us set

$$R = X^2, \quad S = Y^2, \quad T = XY + YX, \quad U = X(XZ + ZX).$$

Lemma 5.1. *In the algebra $S(t_H) \otimes A_{a,b,c}$ we have the following equalities:*

$$\begin{aligned} \mu_\alpha(R) &= at_x^2 u_1, \\ \mu_\alpha(S) &= (at_y^2 + bt_1 t_y + ct_1^2) u_1, \\ \mu_\alpha(T) &= t_x(2at_y + bt_1) u_1, \\ \mu_\alpha(U) &= at_x^2(2t_z + bt_x) u_1. \end{aligned}$$

Proof. This follows from a straightforward computation. Let us compute $\mu_\alpha(S)$ as an example. We have

$$\begin{aligned} \mu_\alpha(S) &= \mu_\alpha(Y)^2 = (t_1 u_y + t_y u_x)^2 \\ &= t_y^2 u_x^2 + t_1 t_y (u_x u_y + u_y u_x) + t_1^2 u_y^2 \\ &= (at_y^2 + bt_1 t_y + ct_1^2) u_1 \end{aligned}$$

in view of the definition of $A_{a,b,c}$. □

We now exhibit two non-trivial H_4 -identities.

Proposition 5.2. *The elements*

$$T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R \quad \text{and} \quad ERZ - RXY - \frac{EU - RT}{2}$$

are H_4 -identities for $A_{a,b,c}$.

Proof. It suffices to check that these two elements are killed by μ_α , which is easily done using Lemma 5.1. □

Since E, R, S, T, U are sent under μ_α to $S(t_H) \otimes u_1$, their images in \mathcal{U}_H^α belong to the center \mathcal{Z}_H^α . We assert that after inverting the elements E and R , all relations in \mathcal{Z}_H^α are consequences of the leftmost relation in Proposition 5.2. More precisely, we have the following (see [2, Thm. 10.3]).

Theorem 5.3. *There is an isomorphism of algebras*

$$\mathcal{Z}_H^\alpha[E^{-1}, R^{-1}] \cong k[E, E^{-1}, R, R^{-1}, S, T, U] / (D_{a,b,c}),$$

where

$$D_{a,b,c} = T^2 - 4RS - \frac{b^2 - 4ac}{a} E^2 R.$$

To prove this theorem, we first check that the generic base algebra \mathcal{B}_H^α (whose generators we know) is generated by $E, E^{-1}, R, R^{-1}, S, T, U$; this implies that \mathcal{B}_H^α is the localization

$$\mathcal{B}_H^\alpha = \mathcal{Z}_H^\alpha[E^{-1}, R^{-1}]$$

of \mathcal{Z}_H^α . In a second step, we establish that all relations between the above-listed generators of \mathcal{B}_H^α follow from the sole relation $D_{a,b,c} = 0$.

Let us now turn to the universal comodule algebra \mathcal{U}_H^α . By Proposition 5.2, we have the following relation in \mathcal{U}_H^α , where we keep the same notation for the elements of $T(X_H)$ and their images in \mathcal{U}_H^α :

$$(ER)Z = (R)XY + \left(\frac{EU - RT}{2} \right) \quad \text{in } \mathcal{U}_H^\alpha.$$

The elements in parentheses being central, it follows from the previous relation that if we again invert the central elements E and R , then Z is a linear combination of 1 and XY with coefficients in $\mathcal{B}_H^\alpha = \mathcal{Z}_H^\alpha[E^{-1}, R^{-1}]$. Noting that

$$YX = -XY + T \in -XY + \mathcal{Z}_H^\alpha \subset -XY + \mathcal{B}_H^\alpha,$$

we easily deduce that after inverting E and R any element of \mathcal{U}_H^α is a linear combination of 1, X, Y, XY over \mathcal{B}_H^α .

In [2] the following more precise result was established (see *loc. cit.*, Thm. 10.7). It answers positively the question of § 2.4.

Theorem 5.4. *The localized algebra $\mathcal{U}_H^\alpha[E^{-1}, R^{-1}]$ is free of rank 4 over its center $\mathcal{B}_H^\alpha = \mathcal{Z}_H^\alpha[E^{-1}, R^{-1}]$, and there is an isomorphism of algebras*

$$\mathcal{U}_H^\alpha[E^{-1}, R^{-1}] \cong \mathcal{B}_H^\alpha \langle \xi, \eta \rangle / (\xi^2 - R, \xi\eta + \eta\xi - T, \eta^2 - S).$$

Note that the algebra \mathcal{B}_H^α coincides with the subalgebra of coinvariants of $\mathcal{U}_H^\alpha[E^{-1}, R^{-1}]$.

5.3. An open problem. To complete this survey, we state a problem who will hopefully attract the attention of some researchers.

Fix an integer $n \geq 2$ and suppose that the ground field k contains a primitive n -th root q of 1. Consider the Taft algebra H_{n^2} , which is the algebra generated by two elements x, y subject to the relations

$$x^n = 1, \quad yx = qxy, \quad y^n = 0.$$

This is a Hopf algebra of dimension n^2 with coproduct determined by

$$\Delta(x) = x \otimes x \quad \text{and} \quad \Delta(y) = 1 \otimes y + y \otimes x.$$

The twisted comodule algebras ${}^\alpha H_{n^2}$ have been classified in [7, 13]. (All two-cocycles of H_{n^2} are non-degenerate.)

Give a presentation by generators and relations of the generic base algebra $\mathcal{B}_{H_{n^2}}^\alpha$ and show that $\mathcal{B}_{H_{n^2}}^\alpha$ is a localization of $\mathcal{Z}_{H_{n^2}}^\alpha$. (By [11, Rem. 2.4 (c)] it is enough to consider the case where α is the trivial cocycle.)

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